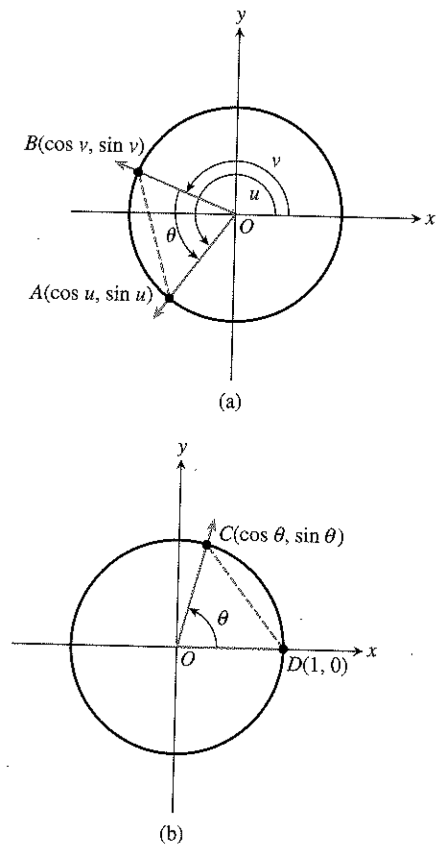


FIGURE 5.9 Angles u and v are in standard position in (a), while angle $\theta = u - v$ is in standard position in (b). The chords shown in the two circles are equal in length.

Square both sides to eliminate the radical and expand the binomials to get

$$\cos^2 u - 2\cos u \cos v + \cos^2 v + \sin^2 u - 2\sin u \sin v + \sin^2 v$$
$$= \cos^2\theta - 2\cos\theta + 1 + \sin^2\theta$$
$$(\cos^2 u + \sin^2 u) + (\cos^2 v + \sin^2 v) - 2\cos u \cos v - 2\sin u \sin v$$
$$= (\cos^2\theta + \sin^2\theta) + 1 - 2\cos\theta$$
$$2 - 2\cos u \cos v - 2\sin u \sin v = 2 - 2\cos\theta$$
$$\cos u \cos v + \sin u \sin v = \cos\theta$$

Finally, since $\theta = u - v$, we can write

$$\cos(u - v) = \cos u \cos v + \sin u \sin v.$$

EXAMPLE 1 Using the Cosine-of-a-Difference Identity

Find the exact value of $\cos 15°$ without using a calculator.

SOLUTION The trick is to write $\cos 15°$ as $\cos(45° - 30°)$; then we can use our knowledge of the special angles.

$$\cos 15° = \cos(45° - 30°)$$
$$= \cos 45° \cos 30° + \sin 45° \sin 30° \quad \text{Cosine difference identity}$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Now try Exercise 5.

Cosine of a Sum

Now that we have the formula for the cosine of a difference, we can get the formula for the cosine of a sum almost for free by using the odd-even identities.

$$\cos(u + v) = \cos(u - (-v))$$
$$= \cos u \cos(-v) + \sin u \sin(-v) \quad \text{Cosine difference identity}$$
$$= \cos u \cos v + \sin u(-\sin v) \quad \text{Odd-even identities}$$
$$= \cos u \cos v - \sin u \sin v$$

We can combine the sum and difference formulas for cosine as follows:

Cosine of a Sum or Difference

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

(Note the sign switch in either case.)

We pointed out in Section 5.1 that the Cofunction Identities would be easier to prove with the results of Section 5.3. Here is what we mean.

EXAMPLE 2 Confirming Cofunction Identities

Prove the identities **(a)** $\cos((\pi/2) - x) = \sin x$ and **(b)** $\sin((\pi/2) - x) = \cos x$.

SOLUTION

(a) $$\cos\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2}\right)\cos x + \sin\left(\frac{\pi}{2}\right)\sin x \quad \text{Cosine sum identity}$$
$$= 0 \cdot \cos x + 1 \cdot \sin x$$
$$= \sin x$$

(b) $$\sin\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) \quad \sin\theta = \cos((\pi/2) - \theta) \text{ by previous proof}$$
$$= \cos(0 + x)$$
$$= \cos x$$

Now try Exercise 41.

Sine of a Difference or Sum

We can use the cofunction identities in Example 2 to get the formula for the sine of a sum from the formula for the cosine of a difference.

$$\sin(u + v) = \cos\left(\frac{\pi}{2} - (u + v)\right) \quad \text{Cofunction identity}$$
$$= \cos\left(\left(\frac{\pi}{2} - u\right) - v\right) \quad \text{A little algebra}$$
$$= \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v \quad \text{Cosine difference identity}$$
$$= \sin u \cos v + \cos u \sin v \quad \text{Cofunction identities}$$

Then we can use the odd-even identities to get the formula for the sine of a difference from the formula for the sine of a sum.

$$\sin(u - v) = \sin(u + (-v)) \quad \text{A little algebra}$$
$$= \sin u \cos(-v) + \cos u \sin(-v) \quad \text{Sine sum identity}$$
$$= \sin u \cos v + \cos u(-\sin v) \quad \text{Odd-even identities}$$
$$= \sin u \cos v - \cos u \sin v$$

We can combine the sum and difference formulas for sine as follows:

Sine of a Sum or Difference

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

(Note that the sign does *not* switch in either case.)

We can solve for a as follows:

$$(a \cos c)^2 + (a \sin c)^2 = 2^2 + 5^2$$
$$a^2 \cos^2 c + a^2 \sin^2 c = 29$$
$$a^2(\cos^2 c + \sin^2 c) = 29$$
$$a^2 = 29 \qquad \text{Pythagorean identity}$$
$$a = \pm\sqrt{29}$$

If we choose a to be positive, then $\cos c = 2/\sqrt{29}$ and $\sin c = 5/\sqrt{29}$. We can identify an acute angle c with those specifications as either $\cos^{-1}(2/\sqrt{29})$ or $\sin^{-1}(5/\sqrt{29})$, which are equal. So, an exact sinusoid for f is

$$f(x) = 2 \sin x + 5 \cos x$$
$$= a \sin (bx + c)$$
$$= \sqrt{29} \sin (x + \cos^{-1}(2/\sqrt{29})) \text{ or } \sqrt{29} \sin (x + \sin^{-1}(5/\sqrt{29}))$$

Now try Exercise 43.

QUICK REVIEW 5.3 *(For help, go to Sections 4.2 and 5.1.)*

In Exercises 1–6, express the angle as a sum or difference of special angles (multiples of 30°, 45°, $\pi/6$, or $\pi/4$). Answers are not unique.

1. 15° **2.** 75°
3. 165° **4.** $\pi/12$
5. $5\pi/12$ **6.** $7\pi/12$

In Exercises 7–10, tell whether or not the identity $f(x + y) = f(x) + f(y)$ holds for the function f.

7. $f(x) = \ln x$ **8.** $f(x) = e^x$
9. $f(x) = 32x$ **10.** $f(x) = x + 10$

SECTION 5.3 EXERCISES

In Exercises 1–10, use a sum or difference identity to find an exact value.

1. $\sin 15°$ **2.** $\tan 15°$
3. $\sin 75°$ **4.** $\cos 75°$
5. $\cos \dfrac{\pi}{12}$ **6.** $\sin \dfrac{7\pi}{12}$
7. $\tan \dfrac{5\pi}{12}$ **8.** $\tan \dfrac{11\pi}{12}$
9. $\cos \dfrac{7\pi}{12}$ **10.** $\sin \dfrac{-\pi}{12}$

In Exercises 11–22, write the expression as the sine, cosine, or tangent of an angle.

11. $\sin 42° \cos 17° - \cos 42° \sin 17°$

12. $\cos 94° \cos 18° + \sin 94° \sin 18°$

13. $\sin \dfrac{\pi}{5} \cos \dfrac{\pi}{2} + \sin \dfrac{\pi}{2} \cos \dfrac{\pi}{5}$

14. $\sin \dfrac{\pi}{3} \cos \dfrac{\pi}{7} - \sin \dfrac{\pi}{7} \cos \dfrac{\pi}{3}$

15. $\dfrac{\tan 19° + \tan 47°}{1 - \tan 19° \tan 47°}$

16. $\dfrac{\tan (\pi/5) - \tan (\pi/3)}{1 + \tan (\pi/5) \tan (\pi/3)}$

17. $\cos \dfrac{\pi}{7} \cos x + \sin \dfrac{\pi}{7} \sin x$ **18.** $\cos x \cos \dfrac{\pi}{7} - \sin x \sin \dfrac{\pi}{7}$

19. $\sin 3x \cos x - \cos 3x \sin x$

20. $\cos 7y \cos 3y - \sin 7y \sin 3y$

21. $\dfrac{\tan 2y + \tan 3x}{1 - \tan 2y \tan 3x}$

22. $\dfrac{\tan 3\alpha - \tan 2\beta}{1 + \tan 3\alpha \tan 2\beta}$

In Exercises 23–30, prove the identity.

23. $\sin \left(x - \dfrac{\pi}{2}\right) = -\cos x$ **24.** $\tan \left(x - \dfrac{\pi}{2}\right) = -\cot x$

25. $\cos \left(x - \dfrac{\pi}{2}\right) = \sin x$

26. $\cos \left[\left(\dfrac{\pi}{2} - x\right) - y\right] = \sin (x + y)$

27. $\sin \left(x + \dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2} \sin x + \dfrac{1}{2} \cos x$

28. $\cos \left(x - \dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2} (\cos x + \sin x)$

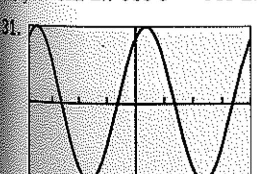
29. $\tan \left(\theta + \dfrac{\pi}{4}\right) = \dfrac{1 + \tan \theta}{1 - \tan \theta}$

30. $\cos \left(\theta + \dfrac{\pi}{2}\right) = -\sin \theta$

In Exercises 31–34, match each graph with a pair of the following equations. Use your knowledge of identities and transformations, not your grapher.

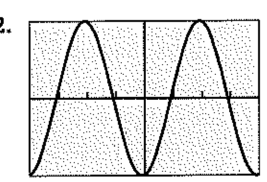
(a) $y = \cos (3 - 2x)$
(b) $y = \sin x \cos 1 + \cos x \sin 1$
(c) $y = \cos (x - 3)$
(d) $y = \sin (2x - 5)$
(e) $y = \cos x \cos 3 + \sin x \sin 3$
(f) $y = \sin (x + 1)$
(g) $y = \cos 3 \cos 2x + \sin 3 \sin 2x$
(h) $y = \sin 2x \cos 5 - \cos 2x \sin 5$

31.



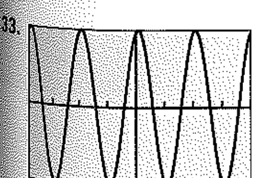
$[-2\pi, 2\pi]$ by $[-1, 1]$

32.



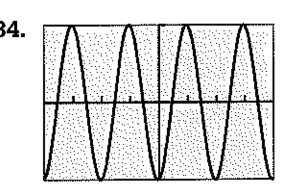
$[-2\pi, 2\pi]$ by $[-1, 1]$

33.



$[-2\pi, 2\pi]$ by $[-1, 1]$

34.



$[-2\pi, 2\pi]$ by $[-1, 1]$

In Exercises 35 and 36, use sum or difference identities (and not your grapher) to solve the equation exactly.

35. $\sin 2x \cos x = \cos 2x \sin x$ **36.** $\cos 3x \cos x = \sin 3x \sin x$

In Exercises 37–42, prove the reduction formula.

37. $\sin \left(\dfrac{\pi}{2} - u\right) = \cos u$ **38.** $\tan \left(\dfrac{\pi}{2} - u\right) = \cot u$

39. $\cot \left(\dfrac{\pi}{2} - u\right) = \tan u$ **40.** $\sec \left(\dfrac{\pi}{2} - u\right) = \csc u$

41. $\csc \left(\dfrac{\pi}{2} - u\right) = \sec u$ **42.** $\cos \left(x + \dfrac{\pi}{2}\right) = -\sin x$

In Exercises 43–46, express the function as a sinusoid in the form $y = a \sin (bx + c)$.

43. $y = 3 \sin x + 4 \cos x$ **44.** $y = 5 \sin x - 12 \cos x$
45. $y = \cos 3x + 2 \sin 3x$ **46.** $y = 3 \cos 2x - 2 \sin 2x$

In Exercises 47–55, prove the identity.

47. $\sin (x - y) + \sin (x + y) = 2 \sin x \cos y$
48. $\cos (x - y) + \cos (x + y) = 2 \cos x \cos y$
49. $\cos 3x = \cos^3 x - 3 \sin^2 x \cos x$
50. $\sin 3u = 3 \cos^2 u \sin u - \sin^3 u$
51. $\cos 3x + \cos x = 2 \cos 2x \cos x$
52. $\sin 4x + \sin 2x = 2 \sin 3x \cos x$
53. $\tan (x + y) \tan (x - y) = \dfrac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \tan^2 y}$
54. $\tan 5u \tan 3u = \dfrac{\tan^2 4u - \tan^2 u}{1 - \tan^2 4u \tan^2 u}$
55. $\dfrac{\sin (x + y)}{\sin (x - y)} = \dfrac{(\tan x + \tan y)}{(\tan x - \tan y)}$

Standardized Test Questions

56. True or False If A and B are supplementary angles, then $\cos A + \cos B = 0$. Justify your answer.

57. True or False If $\cos A + \cos B = 0$, then A and B are supplementary angles. Justify your answer.

You should answer these questions without using a calculator.

58. Multiple Choice If $\cos A \cos B = \sin A \sin B$, then $\cos (A + B) =$

(A) 0 **(B)** 1
(C) $\cos A + \cos B$ **(D)** $\cos B + \cos A$
(E) $\cos A \cos B + \sin A \sin B$